Rayleigh number was higher than the transition Rayleigh number given by Powe *et al.* [6]. However, as other work [4] suggests that this phenomenon is not two-dimensional but three-dimensional, it presumably is necessary to perform the transient 3-D analysis to simulate it.

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# Deflection of a circular jet by a weak cross-flow

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## NOMENCLATURE

| | |
|---|---|
| $b$ | transverse length scale (also used as length scale for other directions) |
| $f$ | function |
| $g$ | function |
| $I$ | constant |
| $K$ | numerical constant |
| $U$ | ambient longitudinal velocity |
| $U_0$ | initial jet velocity |
| $u$ | longitudinal velocity |
| $u_m$ | maximum jet velocity at a section |
| $V$ | ambient lateral velocity |
| $v$ | lateral velocity |
| $w$ | vertical velocity |
| $x$ | longitudinal coordinate |
| $y$ | lateral coordinate |
| $z$ | vertical coordinate. |

Greek symbols

| | |
|---|---|
| $\alpha$ | ratio of initial jet velocity to cross-stream velocity |
| $\zeta$ | non-dimensional vertical coordinate |
| $\eta$ | non-dimensional lateral coordinate |
| $\theta$ | momentum thickness |
| $\rho$ | fluid density |
| $\tau$ | shear stress. |

## INTRODUCTION

THE PROBLEM of predicting the trajectory of a jet injected into a moving ambient fluid is of much importance and has been extensively studied both theoretically and experimentally. The authors have recently developed a simple method for the calculation of the deflection of a plane non-buoyant jet issuing into a weak uniform cross-flow [1, 2]. The objective of this study is to extend this method to the much more complex and practical case of a circular jet in a moving ambient flow.

The calculation method is based on a similarity analysis of the axial momentum equation. The special case of the trajectory of a circular jet in a simple cross-flow will be compared with some of the available experimental results.

## CIRCULAR JET IN A GENERAL CROSS-FLOW

Figure 1 shows a circular jet issuing from a nozzle of radius $r_0$ with a velocity of $U_0$ and exposed to a cross-flow of velocity $V$ (which may be a function of $x$) and a uniform flow $U$ in the axial direction. The trajectory is given by $\bar{y}(x)$. The continuity equation and the equation of motion for the deflected jet are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{1}{\rho}\left(\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial x}\right) \tag{2}$$

where $u$, $v$, $w$ are the time-averaged velocities in the $x$, $y$ and $z$ directions and $\tau_{yx}$ and $\tau_{zx}$ are the turbulent shear stresses acting in the axial direction (the laminal stresses have been neglected). For convenience, the velocities may be considered as a sum of local and background components:

$$u = U+u' \tag{3}$$

$$v = V+v' \tag{4}$$

where the prime denotes a local component. These relations may be introduced into equations (1) and (2) with $u$ and $v$ now representing $u'$ and $v'$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{5}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+U\frac{\partial u}{\partial x}+V\frac{\partial u}{\partial y}=\frac{1}{\rho}\left(\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right). \tag{6}$$

Integrating equation (6) with respect to $z$ first and then $y$ from $-\infty$ to $\infty$, one obtains the results

$$\frac{\mathrm{d}}{\mathrm{d}x}\iint_{-\infty}^{\infty} u(U+u)\,\mathrm{d}y\,\mathrm{d}z = 0. \tag{7}$$

This equation indicates that excess momentum flux is conserved in the $x$ direction. Multiplying equation (6) by $u$ and integrating as in the previous case

$$\frac{\mathrm{d}}{\mathrm{d}x}\iint_{-\infty}^{\infty} u^2(U+u)\,\mathrm{d}y\,\mathrm{d}z = -\frac{2}{\rho}\iint_{-\infty}^{\infty}\left(\tau_{yx}\frac{\partial u}{\partial y}+\tau_{zx}\frac{\partial u}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z. \tag{8}$$

This is the integral energy equation for an axisymmetric jet (written in Cartesian coordinates) with a general cross-flow. It is to be noted that $V$ does not appear in equations (7) and (8). If it is assumed that the length scales are the same in both cross-sectional directions, then these equations can be solved for $u_m$ and $b$, the characteristic scales.

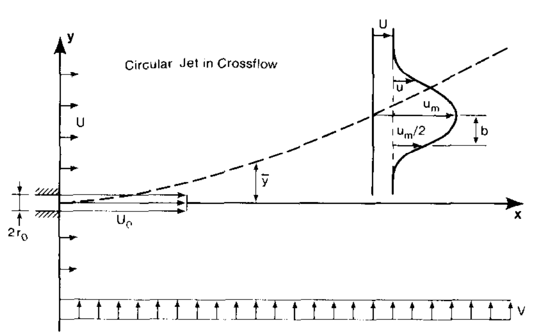


FIG. 1. Definition sketch.

Multiplying equation (6) by $y$ and integrating with respect to $z$ first and then with respect to $y$, from $-\infty$ and $+\infty$ (in both cases), after some simplifications, one obtains the result

$$\frac{\mathrm{d}}{\mathrm{d}x}\iint_{-\infty}^{\infty} y(U+u)\,\mathrm{d}y\,\mathrm{d}z-\iint_{-\infty}^{\infty} uv\,\mathrm{d}y\,\mathrm{d}z-\iint_{-\infty}^{\infty} uV\,\mathrm{d}y\,\mathrm{d}z = -\frac{1}{\rho}\iint_{-\infty}^{\infty}\tau_{yx}\,\mathrm{d}y\,\mathrm{d}z. \tag{9}$$

The shear stress integral on the right-hand side of equation (9) will vanish if a velocity gradient model with a constant momentum diffusivity is assumed. In any event, this term will be small due to approximate antisymmetry of the shear stress distribution. The second term on the left-hand side may also be neglected in comparison to the third term since $v$ is of the same order of magnitude as $V$ (at most about 10% of initial jet velocity $U_0$) but is of opposite sign on either side of the jet axis while $V$ is a constant. In addition $v$ is smallest where $u$ is largest. It is reasonable then to expect that the second integral will be at least one order of magnitude smaller than the third. This approximation is weakest near the jet outlet where $v$ is largest. With these approximations equation (9) will reduce to

$$\frac{\mathrm{d}}{\mathrm{d}x}\iint_{-\infty}^{\infty} y(U+u)\,\mathrm{d}y\,\mathrm{d}z = \iint_{-\infty}^{\infty} uV\,\mathrm{d}y\,\mathrm{d}z. \tag{10}$$

Equation (10) may be interpreted as saying that the rate of change of the flux of moment of momentum is equal to the convection of the $u$ velocity in the $y$ direction.

To proceed further, assume

$$\frac{u}{u_m}=f_1\left(\frac{z}{b}\right)g_1\left(\frac{y-\bar{y}}{b}\right)=f_1(\zeta)g_1(\eta) \tag{11}$$

$$\frac{\tau_{yx}}{\rho u_m^2}=f_2(\zeta)g_2(\eta) \tag{12}$$

$$\frac{\tau_{zx}}{\rho u_m^2}=f_3(\zeta)g_3(\eta) \tag{13}$$

wherein $\zeta = z/b$, $\eta = (y-\bar{y})/b$ and $f_1, f_2, f_3, g_1, g_2$ and $g_3$ denote functions. Substituting equations (11)–(13) into equation (10) and after some simplifications, one obtains the results

$$\frac{\mathrm{d}}{\mathrm{d}x}u_m b^2\iint_{-\infty}^{\infty} f_1 g_1 \bar{y}(U+u_m f_1 g_1)\,\mathrm{d}\eta\,\mathrm{d}\zeta = V u_m b^2\iint_{-\infty}^{\infty} f_1 g_1\,\mathrm{d}\eta\,\mathrm{d}\zeta. \tag{14}$$

Since in the three final equations, equations (7), (8) and (14), equations (7) and (8) are in principle the same as those for the jet in a coflowing stream without a deflecting flow, the scales $u_m$ and $b$ can be taken from the study of Pande and Rajaratnam [3] as

$$\frac{U}{u_m}\simeq\frac{1}{K}\left(\frac{x}{\theta}\right)\quad\text{for}\quad\frac{x}{\theta}<150 \tag{15}$$

where $1/K$ is the slope of the curve in Fig. 6 of reference [2] and $K$ has the approximate value of 12.2

$$b^2=\frac{1}{2}\,\frac{\theta^2}{(u_m/U)^2F_2+(u_m/U)F_1} \tag{16}$$

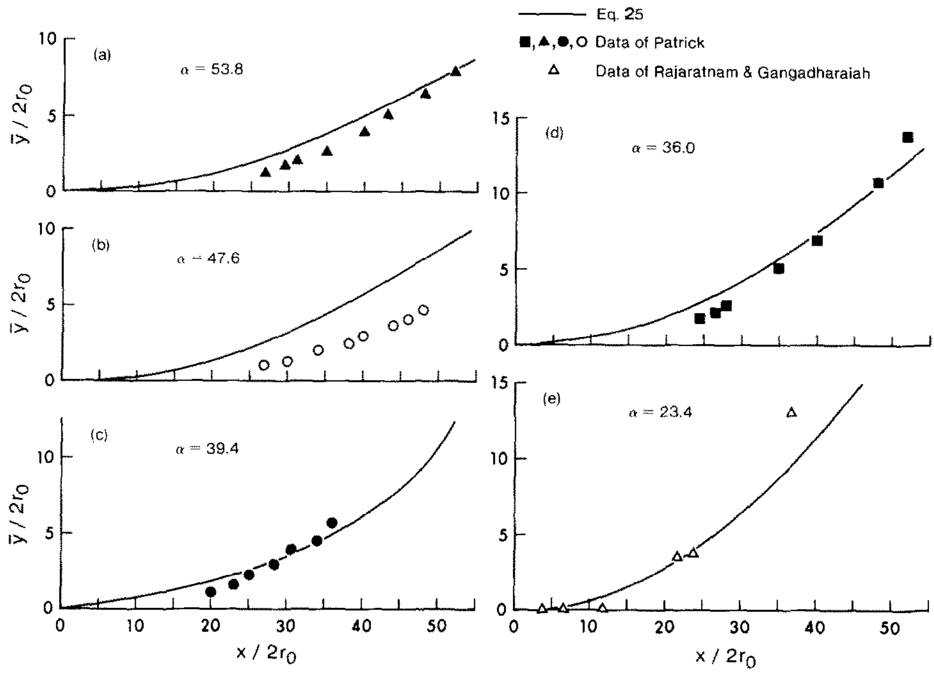


FIG. 2. Circular jets in weak cross-flow. Comparison of equation (25) with experimental results.

where $\theta$ is the momentum thickness given by the equation

$$\theta = r_0\left[\frac{U_0}{U}\left(\frac{U_0}{U}-1\right)\right]^{1/2} \tag{17}$$

$U_0$ being the velocity of the jet at the nozzle and $F_1$ and $F_2$ are integral constants [3]. Equation (14) can be rewritten as

$$\frac{d}{dx}[u_m b^2 \bar{y}(UI_1+u_2I_2)] = Vu_m b^2 I_1 \tag{18}$$

wherein (if exponential functions are utilized for $f_1$ and $g_1$, [4])

$$I_1 = \iint_{-\infty}^{\infty} f_1 g_1 \, d\eta \, d\zeta = \iint_{-\infty}^{\infty} \exp\left[-0.693(\eta^2+\zeta^2)\right] d\eta \, d\zeta \tag{19}$$

$$I_2 = \iint_{-\infty}^{\infty} f_1^2 g_1^2 \, d\eta \, d\zeta = \iint_{-\infty}^{\infty} \exp\left[-1.386(\eta^2+\zeta^2)\right] d\eta \, d\zeta. \tag{20}$$

Using equation (7), equation (18) can be reduced to

$$\frac{d\bar{y}}{dx} = \frac{I_1 V}{(I_1 U + I_2 u_m)}. \tag{21}$$

Using equations (15) and (16) and the result $I_2/I_1 = 1/2$, equation (21) can be reduced to

$$\frac{d\bar{y}}{dx} = \frac{V}{U}\frac{(x/\theta)}{[(x/\theta)+K/2]}. \tag{22}$$

For the case of a uniform cross-flow ($V$ = constant) equation (22) may be integrated using $\bar{y} = 0$ at $x = 0$ to give

$$\frac{\bar{y}}{\theta} = \frac{V}{U}\left[\frac{x}{\theta} - \frac{K}{2}\ln\left\{\frac{(x/\theta + K/2)}{K/2}\right\}\right]. \tag{23}$$

For the case of a simple circular jet in a weak uniform cross-flow, $U = 0$ and with some algebra may be reduced to

$$\frac{d\bar{y}}{dx} = \frac{2Vx}{Kr_0U_0}. \tag{24}$$

Integrating

$$\frac{\bar{y}}{r_0} = \frac{1}{K}\frac{V}{U_0}\left(\frac{x}{r_0}\right)^2. \tag{25}$$

Equation (25) indicates that the deflection $\bar{y}$ is proportional to $x^2$. Figure 2 shows a comparison of equation (25) with some of the (readily available) experimental observations for large values of the ratio $\alpha$ of the jet velocity $U_0$ to the cross-stream velocity $V$. Considering that $K$ was evaluated from rather general considerations (i.e. observations on jets in coflowing streams) the agreement of equation (25) with the experimental results in Fig. 2 should be considered very encouraging.

## CONCLUSIONS

Presented herein is a simple analytical method for calculation of trajectories of circular jets in weak cross-flows. Based on the assumption of general similarity of the jet, a simple convection relation is derived with no experimental constants to evaluate. The result is derived for an asymptotically weak cross-flow but may be applied as a first approximation to many practical problems. Since the present derivations do not assume a uniform cross-flow in the $x$ direction, non-uniform cross-flows could also be included. The agreement of the predicted deflection of circular jets with experimental observations in weak cross-flows has been found to be satisfactory.

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# A Monte Carlo procedure for straight convecting boundaries

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## NOMENCLATURE

$a$ coefficient
$A$ upper limit of integration, $A = \int_{-\pi/2}^{\pi/2} P(z)\,dz$
$b$ coefficient
$B$ coefficient,
$$B = \int_{-\pi/2}^{\pi/2} [1-(B_i/3)\cos(3z)]P(z)\,dz$$
$B_i$ Biot number, $B_i = hR/k$
$C$ constant
$d_{ave}$ average distance reflected into interior perpendicular to boundary
e natural number, e = 2.71828...
$E$ sum of squared errors, see equation (16)
$f_n$ function of $z$, $f_n(z)$
$$= \cos(nz)+B_i\cos[(n+1)z]/(n+1),\ n \text{ even}$$
$$= \sin(nz)+B_i\sin[(n+1)z]/(n+1),\ n \text{ odd}$$
$\overline{f_n^2}$ function of $B_i, \overline{f_n^2}$
$$= \pi+4B_i+\pi B_i^2/2,\ n=0$$
$$= \pi/2+4B_i/(2n+1)+\pi B_i^2/[2(n+1)^2],\ n \neq 0$$
$F$ cumulative distribution function,
$$F = \int_{-\pi/2}^{z} P(z')\,dz'/A$$
$G$ function of $x$, see equation (10)
$h$ heat transfer coefficient
$k$ thermal conductivity
$L$ distance reflected in one-dimensional approximation
$L_1, L_2$ differential operators, see equation (10)
$N$ random number
$P$ probability density function, see equation (17)
$q$ negative derivative of $T$ with respect to $z$, $q = -\partial T/\partial z$
$q''$ heat release rate per unit area
$q_0$ function of $x$, see equation (13)
$Q$ dimensionless heat release rate parameter, $Q = q''R^2/4k$
$r$ distance from the vertex
$R$ straight side length
$T$ temperature
$T_{boundary}$ boundary temperature
$T_{interior}$ temperature at distance $L$ into interior
$T_0$ function of $x$, see equation (12)
$T_p$ peripheral temperature
$T_v$ vertex temperature, $T_v = T(x \to \infty)$
$T_\infty$ environmental temperature
$x$ dimensionless coordinate, $x = \ln(R/r)$
$y$ temperature parameter, $y = T_p(z)-T_\infty+Q[1-(B_i/3)\cos(3z)]$
$z$ dimensionless coordinate, $z = \theta-\theta_m/2$.

Greek symbols
$\alpha$ differential operator, $\alpha = \partial(\ )/\partial x$
$\theta$ angle from the straight side
$\theta_m$ included angle
$\zeta$ differential operator, $\zeta = \partial(\ )/\partial z$
$\pi$ natural number, $\pi = 3.14159\ldots$

Subscripts
$m$ index
$n$ index.

## INTRODUCTION

THE FLOATING random walk Monte Carlo method for two-dimensional steady conduction requires a procedure to determine the next move of a walker situated on a boundary that convectively exchanges heat with an environment at known temperature across a known heat transfer coefficient. Previously [1], a one-dimensional finite difference approximation to the convective boundary condition has been employed in the direction perpendicular to the boundary as

$$T_{boundary} = [T_{interior}+(hL/k)T_\infty]/[1+hL/k].$$

Such an approximation requires the reflective move back into the solid to be small to preserve accuracy and to be perpendicular to the boundary. A two-dimensional approximation is subject to the same difficulties.

An amelioration will be devised from a solution for the vertex temperature of a sector of a circle experiencing convection along its straight sides and with a specified temperature along the circular arc, the vertex being on the convective boundary.